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Improved dispersive analysis of the scalar form factor of the nucleon

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We present a coupled system of integral equations for the $\pi\pi \to \bar{N}N$ and $\bar{K}K \to \bar{N}N$ S-waves derived from Roy–Steiner equations for pion–nucleon scattering. We discuss the solution of the corresponding two-channel Muskhelishvili–Omnès problem and apply the results to a dispersive analysis of the scalar form factor of the nucleon fully including $\bar{K}K$ intermediate states. In particular, we determine the corrections Δ_σ and Δ_D, which are needed for the extraction of the pion–nucleon σ term from πN scattering, and show that the difference $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)\,\text{MeV}$ is insensitive to the input πN parameters.

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1. Introduction

The pion–nucleon σ term $\sigma_{\pi N}$ measures the contribution of the light quarks to the nucleon mass m, and is directly related to the form factor of the scalar current

$$\sigma(t) = \frac{1}{2m}\langle N(p')|\hat{m}(\bar{u}u+\bar{d}d)|N(p)\rangle, \qquad t=(p'-p)^2, \qquad \hat{m}=\frac{m_{\mathrm{u}}+m_{\mathrm{d}}}{2}, \tag{1.1}$$

at vanishing momentum transfer $\sigma(0)=\sigma_{\pi N}$. The standard procedure for its extraction from pion–nucleon (πN) scattering relies on the low-energy theorem [1, 2]

$$F_\pi^2 \bar{D}^+\big(s=m^2, t=2M_\pi^2\big) = \sigma\big(2M_\pi^2\big) + \Delta_{\mathrm{R}}, \tag{1.2}$$

which relates the Born-term-subtracted isoscalar πN scattering amplitude at the Cheng–Dashen point $\bar{D}^+(s=m^2, t=2M_\pi^2)$ to the scalar form factor evaluated at $2M_\pi^2$. The remainder Δ_{R} is free of chiral logarithms at full one-loop order in chiral perturbation theory (ChPT) [3, 4], and has been estimated as [3]

$$|\Delta_{\mathrm{R}}| \lesssim 2\,\mathrm{MeV}. \tag{1.3}$$

Rewriting (1.2) in terms of

$$\Delta_D = F_\pi^2\Big\{\bar{D}^+\big(s=m^2, t=2M_\pi^2\big) - d_{00}^+ - 2M_\pi^2 d_{01}^+\Big\}, \qquad \Delta_\sigma = \sigma\big(2M_\pi^2\big) - \sigma_{\pi N}, \tag{1.4}$$

the extraction of the σ term reduces to the determination of the subthreshold parameters d_{00}^+ and d_{01}^+ as well as the combination $\Delta_D - \Delta_\sigma - \Delta_{\mathrm{R}}$. The first two corrections can be calculated using a dispersive approach [5]

$$\Delta_D - \Delta_\sigma = (-3.3 \pm 0.2)\,\mathrm{MeV}, \tag{1.5}$$

where the error only covers the uncertainties in the $\pi\pi$ phase shifts available at that time. Here, we update the determination of Δ_D and Δ_σ using modern $\pi\pi$ phases, fully including $\bar{K}K$ intermediate states, and carefully studying the dependence of the results on πN subthreshold parameters as well as the πN coupling constant.

2. Scalar pion and kaon form factors

We first consider the case of the scalar pion and kaon form factors $F_\pi^S(t)$ and $F_K^S(t)$, which serve both to illustrate the method and as input for the scalar form factor of the nucleon. Unitarity in the $\pi\pi/\bar{K}K$ system intertwines both form factors according to [6]

$$\operatorname{Im} F^S(t) = \big(T(t)\big)^* \Sigma(t) F^S(t), \qquad F^S(t) = \begin{pmatrix} F_\pi^S(t) \\ \frac{2}{\sqrt{3}} F_K^S(t) \end{pmatrix}, \tag{2.1}$$

with the phase-space factor

$$\Sigma(t) = \operatorname{diag}\Big(\sigma_t^\pi \theta\big(t-t_\pi\big), \sigma_t^K \theta\big(t-t_K\big)\Big), \qquad \sigma_t^i = \sqrt{1-\frac{t_i}{t}}, \qquad t_i = 4M_i^2 \qquad i\in\{\pi, K\}, \tag{2.2}$$

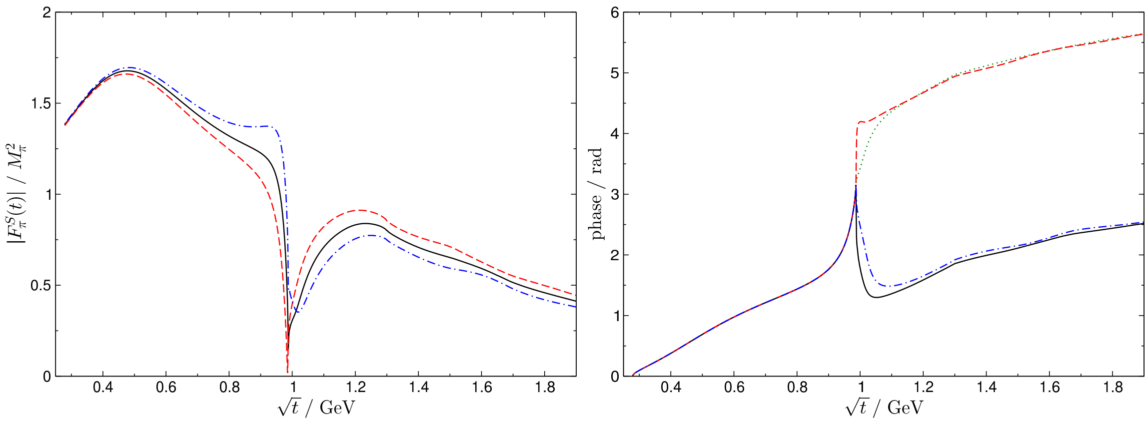

Figure 1: Modulus and phase of the scalar pion form factor. The solid, dashed, and dot-dashed lines refer to $F_K^S(0) = M_\pi^2/2$, $0.4M_\pi^2$, and $0.6M_\pi^2$. The phase of $F_\pi^S(t)$ is compared to δ_0^0, as indicated by the dotted line.

and the T-matrix

$$T(t) = \begin{pmatrix} \frac{\eta_0^0(t)e^{2i\delta_0^0(t)}-1}{2i\sigma_t^\pi} & |g(t)|e^{i\psi_0^0(t)} \\ |g(t)|e^{i\psi_0^0(t)} & \frac{\eta_0^0(t)e^{2i(\psi_0^0(t)-\delta_0^0(t))}-1}{2i\sigma_t^K} \end{pmatrix}, \tag{2.3}$$

expressed in terms of the $\pi\pi$ and $\pi\pi \to \bar{K}K$ phase shifts δ_0^0 and ψ_0^0 as well as the inelasticity parameter $\eta_0^0 = \sqrt{1-4\sigma_t^\pi\sigma_t^K|g(t)|^2\theta(t-t_\pi)}$. The two-channel Muskhelishvili–Omnès (MO) problem [7, 8] defined by the unitarity relation (2.1) permits two linearly independent solutions Ω_1, Ω_2 [7], which may be combined in the Omnès matrix $\Omega(t)$. In general, there is no analytical solution for $\Omega(t)$, we follow here the discretization method of [9] for its numerical calculation.

Since the form factors are devoid of a left-hand cut, they are related directly to the solutions of the MO problem with coefficients determined by $F_\pi^S(0)$ and $F_K^S(0)$ [6]. Using ChPT at $\mathscr{O}(p^4)$ and the low-energy constants from [10] we find

$$F_\pi^S(0) = (0.984 \pm 0.006)M_\pi^2, \qquad F_K^S(0) = (0.4\ldots0.6)M_\pi^2, \tag{2.4}$$

which, together with δ_0^0 and η_0^0 from an extended Roy-equation analysis of $\pi\pi$ scattering [11], ψ_0^0 from partial-wave analyses [12], and $|g(t)|$ from a Roy–Steiner (RS) analysis of πK scattering [13], yield the results for $F_\pi^S(t)$ depicted in Fig. 1. The strong dependence of $F_\pi^S(t)$ near t_K on $F_K^S(0)$ attests to the inherent two-channel nature of the problem and implies that an effective single-channel description in terms of the phase of $F_\pi^S(t)$ only works for sufficiently large $F_K^S(0)$.

3. From Roy–Steiner equations to the scalar form factor

Unitarity couples the $\pi\pi \to \bar{N}N$ and $\bar{K}K \to \bar{N}N$ S-waves $f_+^0(t)$ and $h_+^0(t)$ analogously to (2.1)

$$\text{Im}\, f(t) = \big(T(t)\big)^*\Sigma(t)f(t), \qquad f(t) = \begin{pmatrix} f_+^0(t) \\ \frac{2}{\sqrt{3}}h_+^0(t) \end{pmatrix}, \tag{3.1}$$

but due to the presence of the left-hand cut the solution of the corresponding MO problem involves inhomogeneous contributions, which may be derived from RS equations, cf. [13–15]. Generically,

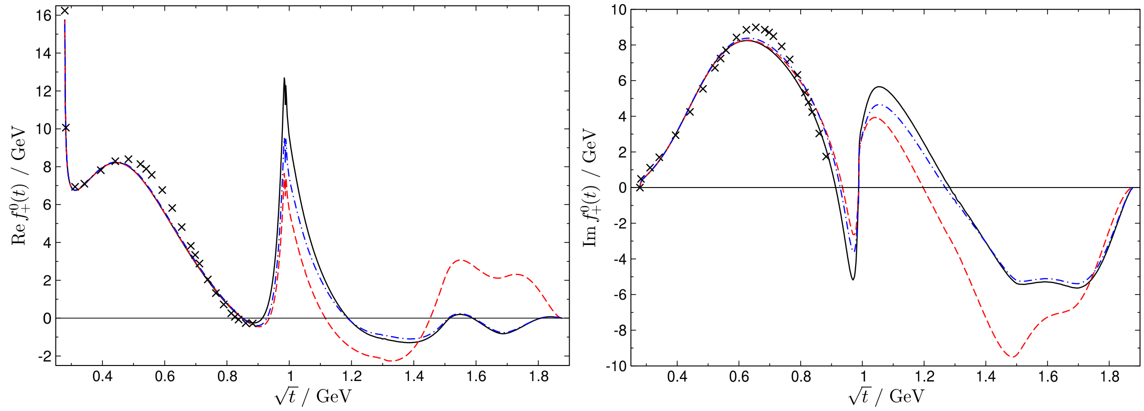

Figure 2: Results for the real and imaginary part of $f_+^0(t)$. The solid, dashed, and dot-dashed lines refer to the input RS1, RS2, and RS3 as described in the main text. The black crosses indicate the results of [17].

the integral equation takes the form

$$f(t) = \Delta(t) + (a+bt)(t-4m^2) + \frac{t^2(t-4m^2)}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\, f(t')}{t'^2(t'-4m^2)(t'-t)}, \qquad (3.2)$$

where $\Delta(t)$ includes Born terms, s-channel integrals, and higher t-channel partial waves, while a and b subsume subthreshold parameters that emerge as subtraction constants. The main difficulty in the evaluation of the formal solution

$$\begin{aligned} f(t) &= \Delta(t) + (t-4m^2)\Omega(t)(\mathbb{1} - t\,\dot{\Omega}(0))a + t(t-4m^2)\Omega(t)b \qquad (3.3)\\ &- \frac{t^2(t-4m^2)}{\pi}\Omega(t)\int_{t_\pi}^{t_\mathrm{m}} \mathrm{d}t' \frac{\mathrm{Im}\,\Omega^{-1}(t')\Delta(t')}{t'^2(t'-4m^2)(t'-t)} + \frac{t^2(t-4m^2)}{\pi}\Omega(t)\int_{t_\mathrm{m}}^{\infty} \mathrm{d}t' \frac{\Omega^{-1}(t')\mathrm{Im}\, f(t')}{t'^2(t'-4m^2)(t'-t)}, \end{aligned}$$

concerns the construction of the Omnès matrix for a finite matching point t_m [14].

In the numerical analysis we put $\mathrm{Im}\, f(t) = 0$ above t_m, which we choose as $t_\mathrm{m} = 4m^2$ (thus exploiting a kinematical zero of $f(t)$), take the πN and KN s-channel partial waves from [16], and use the KH80 πN coupling constant and subthreshold parameters as reference point [17]. In order to assess the uncertainties for higher energies we consider the following variants of the input: first, we keep the phase shifts δ_0^0 and ψ_0^0 constant above $\sqrt{t_0} = 1.3\,\mathrm{GeV}$ ("RS1"), where 4π intermediate states become important and thus the two-channel approximation will break down, and second, guide both phase shifts smoothly to their asymptotic value of 2π as for the meson form factors ("RS2"). Finally, we amend RS1 in such a way that $\Delta_2(t)$, the KN component of the inhomogeneity, is put to zero in order to assess the uncertainty in the KN input ("RS3"). The corresponding results for $f_+^0(t)$ depicted in Fig. 2 show that the largest uncertainty is induced by the high-energy phase shifts.

4. Results

The scalar form factor of the nucleon fulfills the unitarity relation

$$\mathrm{Im}\,\sigma(t) = \frac{2}{4m^2 - t}\left\{\frac{3}{4}\sigma_t^\pi \left(F_\pi^S(t)\right)^* f_+^0(t)\theta\left(t-t_\pi\right) + \sigma_t^K \left(F_K^S(t)\right)^* h_+^0(t)\theta\left(t-t_K\right)\right\}, \qquad (4.1)$$

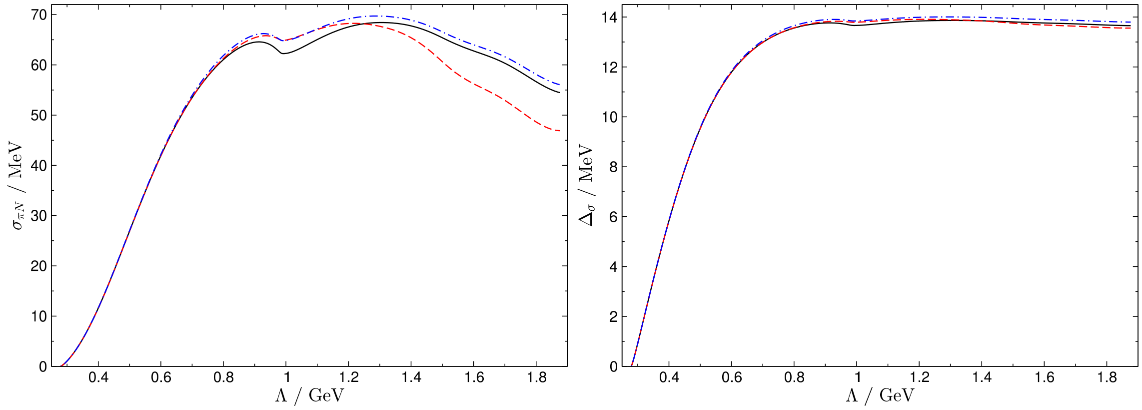

Figure 3: $\sigma_{\pi N}$ and Δ_σ as a function of the integral cutoff Λ.

so that, based on the results of the previous sections, the un- and once-subtracted dispersion relations

$$\sigma(t)=\frac{1}{\pi}\int\limits_{t_\pi}^{\infty}\mathrm{d}t'\frac{\operatorname{Im}\sigma(t')}{t'-t}=\sigma_{\pi N}+\frac{t}{\pi}\int\limits_{t_\pi}^{\infty}\mathrm{d}t'\frac{\operatorname{Im}\sigma(t')}{t'(t'-t)} \tag{4.2}$$

evaluated at $t=0$ and $t=2M_\pi^2$ in principle determine $\sigma_{\pi N}$ and Δ_σ provided the two-channel approximation for the spectral function is sufficiently accurate in the energy range dominating the dispersive integral. Fig. 3 shows that, while the dispersion relation converges too slowly for the σ term itself, the result for Δ_σ becomes stable for $\Lambda \gtrsim 1\,\text{GeV}$. Adding the uncertainties from the spectral function and the variation of the integral cutoff between $\Lambda=1.3\,\text{GeV}$ and $\Lambda=2m$, we find

$$\begin{aligned}\Delta_\sigma &= (13.9\pm 0.3)\,\text{MeV}\\ &\quad+Z_1\left(\frac{g^2}{4\pi}-14.28\right)+Z_2\left(d_{00}^+M_\pi+1.46\right)+Z_3\left(d_{01}^+M_\pi^3-1.14\right)+Z_4\left(b_{00}^+M_\pi^3+3.54\right),\\ Z_1&=0.36\,\text{MeV},\qquad Z_2=0.57\,\text{MeV},\qquad Z_3=12.0\,\text{MeV},\qquad Z_4=-0.81\,\text{MeV},\end{aligned} \tag{4.3}$$

where we have made the dependence on the πN parameters explicit (note that more modern determinations point to lower values of the πN coupling constant around $g^2/4\pi\sim 13.7$ [18 – 20]).

Similarly, the correction Δ_D follows from the t-channel expansion

$$\bar{D}^+(s=m^2,t)=d_{00}^+ + d_{01}^+t-16t^2\int\limits_{t_\pi}^{\infty}\mathrm{d}t'\frac{\operatorname{Im} f_+^0(t')}{t'^2(t'-4m^2)(t'-t)}+\{J\geq 2\}+\{s\text{-channel integrals}\} \tag{4.4}$$

evaluated at $t=2M_\pi^2$, which gives

$$\begin{aligned}\Delta_D &= (12.1\pm 0.3)\,\text{MeV}\\ &\quad+\tilde{Z}_1\left(\frac{g^2}{4\pi}-14.28\right)+\tilde{Z}_2\left(d_{00}^+M_\pi+1.46\right)+\tilde{Z}_3\left(d_{01}^+M_\pi^3-1.14\right)+\tilde{Z}_4\left(b_{00}^+M_\pi^3+3.54\right),\\ \tilde{Z}_1&=0.42\,\text{MeV},\qquad \tilde{Z}_2=0.67\,\text{MeV},\qquad \tilde{Z}_3=12.0\,\text{MeV},\qquad \tilde{Z}_4=-0.77\,\text{MeV}.\end{aligned} \tag{4.5}$$

Comparison with (4.3) shows that the dependence on the πN parameters cancels nearly completely in the difference

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)\,\text{MeV}. \tag{4.6}$$

This cancellation can be explained by the observation that the spectral function in both dispersion relations involves f_+^0 in a very similar manner, so that both integrals are equally affected by the dependence on the πN parameters inherited from f_+^0. In the same way, part of the uncertainties discussed in Sect. 3 drop out, so that the final error estimate for $\Delta_D - \Delta_\sigma$ even decreases compared to the uncertainty in both corrections individually.

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